Subject: Leaving Certificate Maths

Teacher: Mr Murphy

Lesson 17: Calculus III

17.1 Learning Intentions

After this week's lesson you will be able to;

- Explain what is meant by anti-differentiation.
- Anti-differentiate a polynomial function.
- Anti-differentiate an exponential function.
- Find the area under a curve.
- Use the trapezoidal rule.
- Establish the average value of a function.

17.2 Specification

5.2 Calculus	– find first and second derivatives of linear, quadratic and cubic functions by rule – associate derivatives with slopes and tangent lines – apply differentiation to • rates of change • maxima and minima • curve sketching	– differentiate linear and quadratic functions from first principles – differentiate the following functions • polynomial • exponential • trigonometric • rational powers • inverse functions • logarithms – find the derivatives of sums, differences, products, quotients and compositions of functions of the above form – apply the differentiation of above functions to solve problems

17.3 Chief Examiner's Report

B	8	26·6	53	8	Functions/rates of change
B	9	30·7	61	6	Functions/trigonometry/calculus

17.4 Anti-Differentiation

Firstly, we must remember the process of differentiation and what it does.

Differentiation finds a function that allows us to find the value of the slope at a particular point along a curve. However, there are times where we may wish to go in the opposite direction. In other words, go from the derivative back to the function. This process is called **anti-differentiation.**

Differentiation

Function $y = 4x^3$

Derivative $dy/_{dx} = 12x^2$

Anti-Differentiation

So, we have seen that going from a derivative $12x^2$ back to a function $4x^3$. However, this is not the only possible function that yields a derivative of $12x^2$. Our derivative could have originated from many different functions, write down some of your ideas below

Function	**Derivative**
	$12x^2$

So, we have found that if we anti-differentiate a function we cannot say for definite what the value of the constant at the end will be. The constant that we always* add on is **the constant of integration**.

So, to cover all possibilities for our original function we could say that the anti-derivative is $4x^3 + c$.

Anti-Derivative $F(x)$ ← **Anti - Differentiation** ← **Function $f(x)$**

Now we need to look at the mechanics of carrying out anti-differentiation.

To do this we are going to look at $f(x) = x^2$. Our initial thought might be that $F(x) = x^3$. However, if we try to verify our answer by differentiating, we get $3x^2$. As we can see this is 3 times too big, so we need to divide by 3. So, in order to ant-differentiate we follow the below rule:

$$\boxed{\begin{array}{l} f(x) = x^n, \text{then} \\ \\ F(x) = \dfrac{x^{n+1}}{n+1} + c \end{array}}$$

Don't forget to use your tables book to help. For dealing with these topics we can use page 26.

Suimeálaithe | **Integrals**

Tá tairisigh na suimeála fágtha ar lár. | Constants of integration omitted.

$f(x)$	$\int f(x)dx$
$x^n \quad (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
e^{ax}	$\frac{1}{a}e^{ax}$
a^x	$\frac{a^x}{\ln a}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln\lvert\sec x\rvert$

$f(x)$	$\int f(x)dx$
$\cos^2 x$	$\frac{1}{2}\left[x+\frac{1}{2}\sin 2x\right]$
$\sin^2 x$	$\frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$

$f(x)$	$\int f(x)dx$
$\frac{1}{x\sqrt{x^2-a^2}}$	$\frac{1}{a}\sec^{-1}\frac{x}{a}$
$\frac{1}{\sqrt{x^2+a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2+a^2}}{a}\right\rvert$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$
$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert$

Suimeáil na míreanna $\quad \int u\,dv = uv - \int v\,du \quad$ **Integration by parts**

- 26 -

17.5 Integration

So far, by including the constant of integration (+c), we have been dealing with indefinite integrals, i.e. we cannot say for definite what the exact anti-derivative is. Now we will look at the idea of using integrals to find definite integrals. We can use integration to find the area underneath a curve. In other words, the area bounded by a curve (f(x)) and the x-axis.

To help us look at this we will look at a curve: (refer to the video)

So, if we continually increase the number of rectangles, we gain the best estimate of the true area. So, we are going to add up each of the individual areas we can achieve the true area.

Area of a rectangle = height x width

$$= f(x_i) \times \Delta x$$

So, if we are going to do this for each rectangle and add them up, we have:

$$\sum_{i=1}^{n} f(x_i).\Delta x$$

The above means the sum of the area of each rectangle from the first to the last (n).

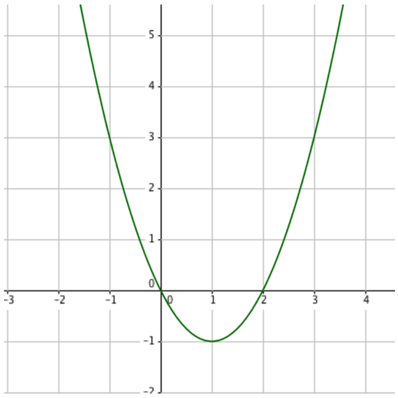
However, as we are going to use infinitely many rectangles, we adopt new notation:

$$Area = \int_a^b f(x)dx$$

So, in the case of our pool face, we have the limits of 0 and 16

$$\int_0^{16} f(x)\,dx$$

Integration Example:



$$f(x) = x^2 - 2$$

To find the area enclosed by the x-axis and f(x):

$$\int_2^3 x^2 - 2\,dx$$

$$\frac{x^3}{3} - 2x\Big|_2^3$$

$$\frac{3^3}{3} - 2(3) - \left(\frac{2^3}{3} - 2(2)\right) = 4.3\ square\ units$$

However, if we look at two different limits, we can notice something different:

$$\int_0^2 x^2 - 2\,dx$$

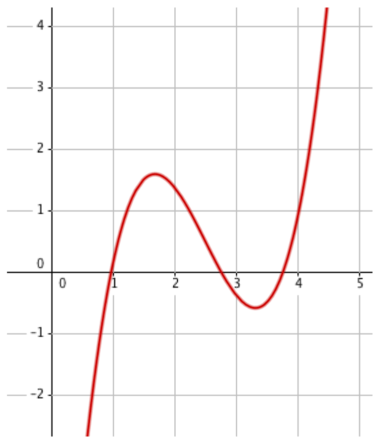
$$\frac{x^3}{3} - 2x\Big|_0^2$$

$$\frac{2^3}{3} - 2(2) - \left(\frac{0^3}{3} - 2(0)\right)$$

Area = **- 1.33** square units

We have got a negative area. Although a negative number does not mean a negative area, it means the region in question is underneath the x-axis.

This idea does however present a problem when the function has an area above and below the x-axis.



This function is an example of two positive areas and one negative. In order to calculate the area bounded by the curve and the x-axis between 1 and 4 we need to:

1) Find the x-coordinates where the curve cuts the x-axis.
2) Split the integral of the entire function up into these 3 regions using the answers to part one as the limits:

$$\int_1^4 f(x)\,dx$$

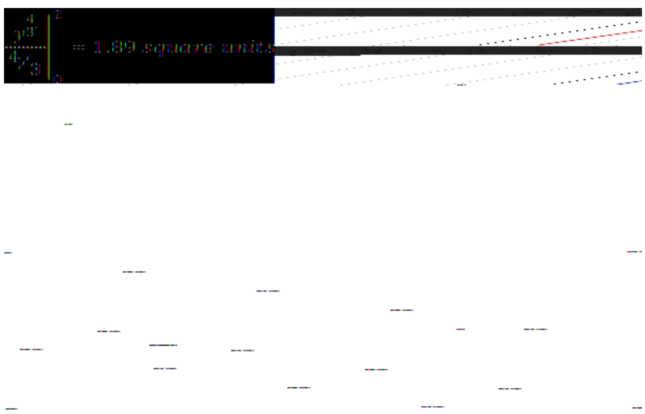
Breaks down into the following:

$$\int_1^{2.8} f(x)dx + \int_{2.8}^{3.8} f(x)\,dx + \int_{3.8}^{4} f(x)\,dx$$

The function in red should be taken as a positive value as it's area will be negative.

17.6 Integration with the y-axis

In some instances, we can be asked to calculate the area between a curve and the y-axis. In which case we must rearrange the function to be in terms of y and uses y-axis values as the limits:



$$y = x^3$$

$$q(x) = x^3$$

$$x = \sqrt[3]{y}$$

$$\int_0^8 \sqrt[3]{y}\, dy$$

$$\int_0^8 y^{\frac{1}{3}}\, dy$$

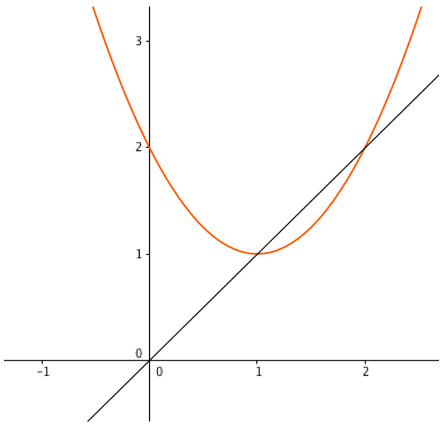
$$\left.\frac{y^{\frac{4}{3}}}{4/3}\right|_0^2 = 1.89\ square\ units$$

This is where we are asked to calculate the area between two functions or curves.

In this case we take each function separately and subtract the areas. Always take the lower function from the higher one.

Make sure to always sketch the functions.

Limits of the integrals are the x-coordinates of the P.O.I.



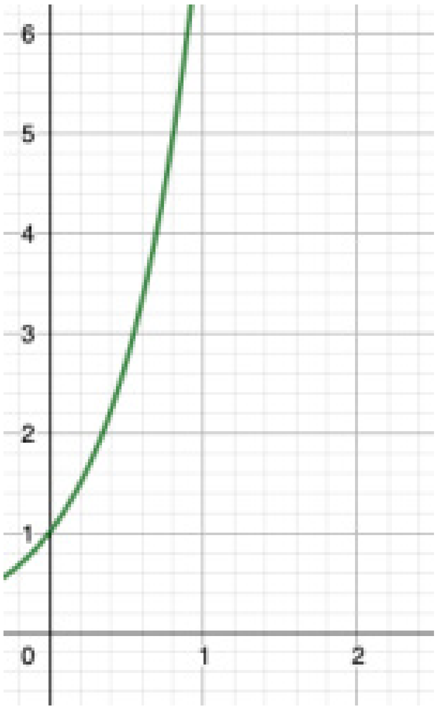
$$\int_a^b f(x)dx - \int_a^b \mathbf{g(x)\ dx}$$

$$\int_1^2 x\, dx - \int_1^2 \mathbf{x^2 - 2x + 2\ dx}$$

$$\text{Area} = \frac{1}{6}\ square\ units$$

17.7 Trapezoidal Rule

This is a rule that was used to estimate area before integration. It involves using Trapezoids instead of rectangles in order to obtain a more accurate estimate of the area under a curve.



$$A = \frac{h}{2}[y_1 + y_n + 2(y_2 + y_3 + y_4 + .. y_{n-1})]$$

$$h = \frac{difference\ in\ limits}{no.\ of\ trapezoids}$$

$$y_1 = e^{2(1)} \quad y_2 = e^{2(1+h)} \quad y_3 = e^{2(1+2h)} \quad y_4 = e^{2(1+3h)}$$

$$A = \frac{1}{6}\left[e^2 + e^4 + 2\left(e^{\frac{8}{3}} + e^{\frac{10}{3}}\right)\right] = \mathbf{24.47\ square\ units}$$

17.8 Average Value of a Function

Sometimes a function can represent quantities such as stock price over a certain time span.

If we know the function, we can use integration to find the average value of the stock over a certain time period

$$Average\ value = \frac{1}{b-a}\int_a^b f(x)\,dx$$

17.9 Recap of the Learning Intentions

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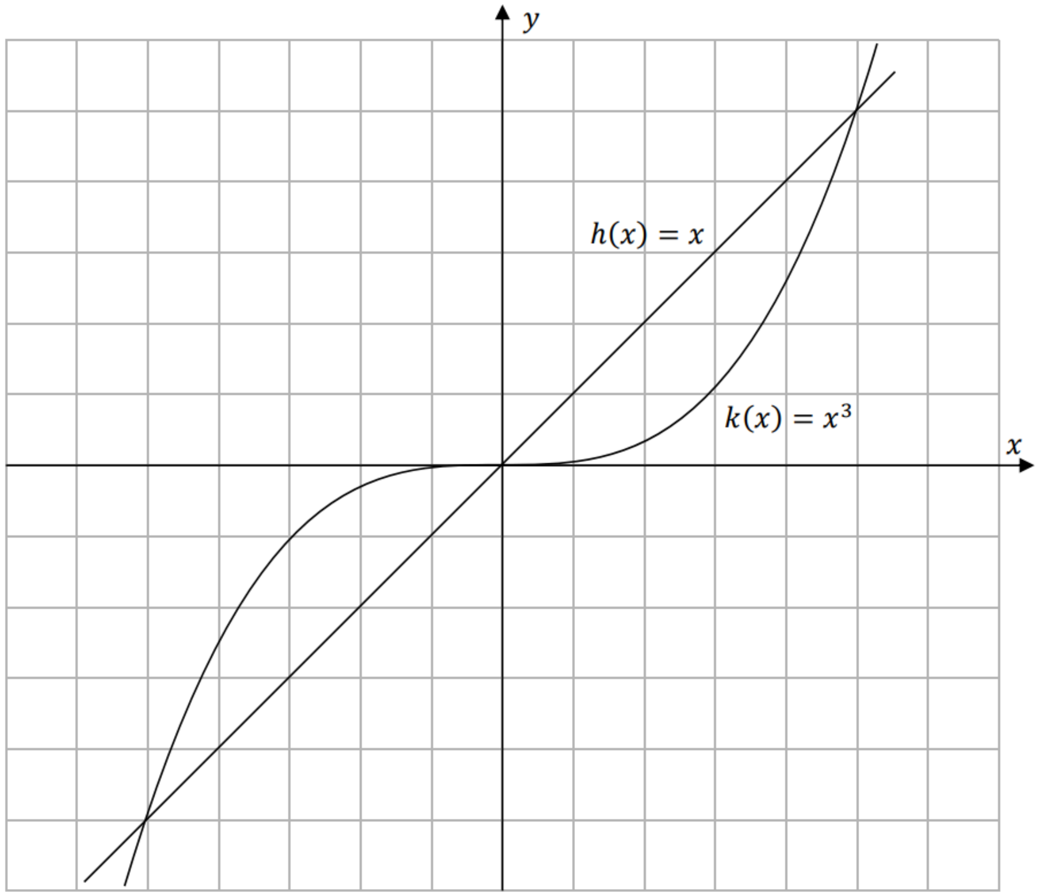
17.10 Homework Task

Parts of the graphs of the functions $h(x) = x$ and $k(x) = x^3$, $x \in \mathbb{R}$, are shown in the diagram below.



(a) Find the co-ordinates of the points of intersection of the graphs of the two functions.

(b) **(i)** Find the total area enclosed between the graphs of the two functions.

(ii) On the diagram on the previous page, using symmetry or otherwise, draw the graph of k^{-1}, the inverse function of k.

17.11 Solutions to 16.10

The weekly revenue produced by a company manufacturing air conditioning units is seasonal. The revenue (in euro) can be approximated by the function:

$$r(t) = 22\,500\cos\left(\frac{\pi}{26}t\right) + 37\,500, \qquad t \geq 0$$

where t is the number of weeks measured from the beginning of July and $\left(\frac{\pi}{26}t\right)$ is in radians.

(a) Find the approximate revenue produced 20 weeks after the beginning of July. Give your answer correct to the nearest euro.

$$r(20) = 22500\cos\left(\frac{\pi}{26}(20)\right) + 37500$$

$$r(20) = 20658.51$$

$$r(20) = €20659$$

(b) Find the two values of the time t, within the first 52 weeks, when the revenue is approximately €26 250.

$$26250 = 22500\cos\left(\frac{\pi}{26}t\right) + 37500$$

$$-11250 = 22500\cos\left(\frac{\pi}{26}t\right)$$

$$-\frac{1}{2} = \cos\left(\frac{\pi}{26}t\right)$$

$$\left(\frac{\pi}{26}t\right) = \left(\frac{2\pi}{3}\right) \qquad \left(\frac{\pi}{26}t\right) = \left(\frac{4\pi}{3}\right)$$

$$t = \left(\frac{52}{3}\right) \qquad t = \left(\frac{104}{3}\right)$$

(c) Find $r'(t)$, the derivative of $r(t) = 22\,500\cos\left(\frac{\pi}{26}t\right) + 37\,500$.

$$r'(t) = 22500\left(-\sin\frac{\pi}{26}t\right)\left(\frac{\pi}{26}t\right)$$

$$r'(t) = \left(-\frac{11250\pi}{13}\right)\left(\sin\frac{\pi}{26}t\right)$$

(d) Use calculus to show that the revenue is increasing 30 weeks after the beginning of July.

$$r'(t) = \left(-\frac{11250\pi}{13}\right)\left(\sin\frac{\pi}{26}t\right)$$

$$r'(30) = \left(-\frac{11250\pi}{13}\right)\left(\sin\frac{\pi}{26}(30)\right)$$

$$r'(t) = 402.164\pi$$

$$r'(t) = 1263.44$$

Which is > 0 therefore increasing